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Two-dimensional intermittent search processes: An alternative to Lévy flight strategies

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Lévy flights are known to be optimal search strategies in the particular case of revisitable targets. In the relevant situation of nonrevisitable targets, we propose an alternative model of two-dimensional (2D) search processes, which explicitly relies on the widely observed intermittent behavior of foraging animals. We show analytically that intermittent strategies can minimize the search time, and therefore do constitute real optimal strategies. We study two representative modes of target detection and determine which features of the search time are robust and do not depend on the specific characteristics of detection mechanisms. In particular, both modes lead to a global minimum of the search time as a function of the typical times spent in each state, for the same optimal duration of the ballistic phase. This last quantity could be a universal feature of 2D intermittent search strategies.

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Search processes, involving a searcher and a target of unknown position, play an important role in many physical, chemical, or biological problems. This is, for instance, the case of reactants diffusing in a solvent until they get close enough to react [1], or of a protein searching for its specific target site on deoxyribose nucleic acid (DNA) [2–4]. One can also mention animals searching for food [5–12] or coast guards trying to locate wreck victims [13]. In all these examples, it is of great importance to minimize the search time. Since the pioneering works [7–9], the question of determining optimal search strategies has been given growing attention [10,11,14–19].

In this context, Lévy flight strategies have been proved to play a crucial role in such optimization problems [9–11]. However, two limitations of these strategies have to be mentioned. First, Lévy flight trajectories have been shown to optimize the search efficiency, but only in the particular case where the targets are regenerated at the same location after a finite time [9,11], which cannot be taken as a general rule. Indeed, in the case of a destructive search where each target can be found only once, or in the case of a single target, the optimal strategy proposed in [9] is not anymore of Lévy type, but reduces to a linear ballistic motion. Second, as for the applications to behavioral ecology, the destructive search is relevant to many situations [5,6]. However, the purely ballistic strategy predicted by [9] in that case cannot account for the generally observed reoriented animal trajectories [5].

Alternatively to these Lévy strategies, it has been observed that intermittent search strategies are widely used by foraging animals [6,20]. Many searchers combine phases of fast displacement, nonreceptive to the targets, and slow reactive search phases. Everyday-life examples also confirm that we instinctively adopt such intermittent behavior when looking for a lost object; we search carefully around one location, then move quickly to another unvisited area and then search again.

Up to now, only one-dimensional (1D) models of such intermittent search have been developed, providing a satisfactory agreement with experimental data from behavioral ecology [12]. Here we develop a model of two-dimensional (2D) intermittent search strategies, which encompasses a much broader field of applications, in particular for animal or human searchers. We show that 2D intermittent search strategies *do optimize* the search time for nonrevisitable targets. We explicitly determine how to share the time between the phases of nonreactive displacement and of reactive search, in order to find a target in the quickest way. From a technical point of view, we also obtain, as a by-product, the mean-first-passage time for a Pearson-type random walk, which belongs to a class of nontrivial problems which have been investigated for a long time [21–25]. Our approach relies on an approximate analytical solution based on a decoupling hypothesis, which proves to reproduce quantitatively our numerical simulations over a wide range of parameters.

Following [12], we consider a two-state searcher (see Fig. 1) of position **r** that performs slow reactive phases (denoted as phases 1), randomly interrupted by fast relocating ballistic flights of constant velocity v and random direction (phases 2). We assume the duration of each phase i to be exponentially distributed with mean τ_i. As fast motion usually strongly degrades perception abilities [6,20], we consider that the searcher is able to find a target only during reactive phases 1. The detection phase involves complex biological processes that we do not aim at modeling accurately here. However, essentially two modes of detection can be put forward, and lead to distinguish between two types of reactive phases 1. The first one, referred to in the following as the "dynamic mode," corresponds to a diffusive modeling (with diffusion coefficient D) of the search phase as recently proposed in [12] in agreement with observations for vision [26], tactile sense, or olfaction [5]. The detection is assumed to be infinitely efficient in this mode; a target is found as soon as the searcher-target distance is smaller than the reaction radius a. On the contrary, in the second mode, denoted as the "static mode," the reaction takes place with a finite rate k, but the searcher is immobile during search phases. Note that this description is commonly adopted in reaction-diffusion systems [1] or operational research [13]. A more realistic description is obtained by combining both modes and considering a diffusive searcher with diffusion coefficient D and finite reaction rate k. In order to reduce the number of parameters and to extract the main features of each mode, we

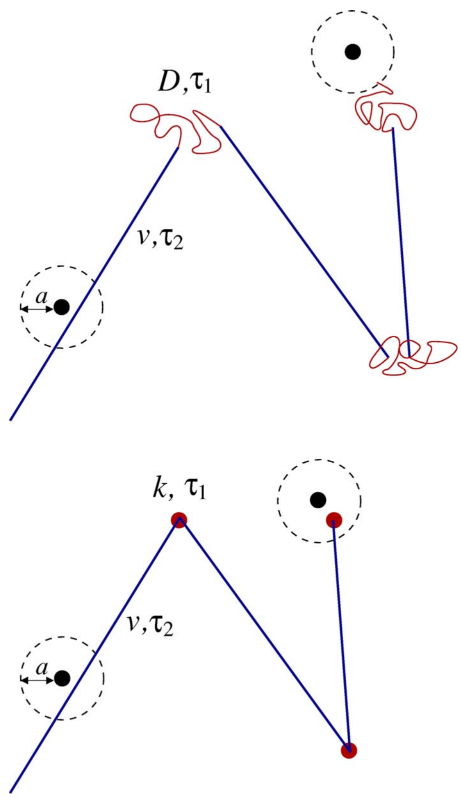

FIG. 1. (Color online) Two models of intermittent search: The searcher alternates slow reactive phases (regime 1) of mean duration τ_1, and fast nonreactive ballistic phases (regime 2) of mean duration τ_2. Top: the slow reactive phase is diffusive and detection is infinitely efficient. Bottom: the slow reactive phase is static and detection takes place with finite rate k.

study them separately by taking successively the limits $k \to \infty$ and $D \to 0$ of this general case. More precisely, in these two limiting cases, we address the following questions: what is the mean time it takes the searcher to find a target? Can this search time be minimized? And if so, for which values of the average durations τ_i of each phase?

We now present the basic equations combining the two search modes introduced above in the case of a pointlike target centered in a spherical domain of radius b with reflecting boundary. Note that this geometry mimics both relevant situations of a single target and of infinitely many regularly spaced nonrevisitable targets. For this process, the mean first passage time (MFPT) at a target satisfies the following backward equation [27]:

$$D\nabla_{\mathbf{r}}^2 t_1 + \frac{1}{2\pi\tau_1}\int_0^{2\pi}(t_2 - t_1)d\theta_{\mathbf{v}} - k\mathbf{I}_a(\mathbf{r})t_1 = -1, \tag{1}$$

$$\mathbf{v}\cdot\nabla_{\mathbf{r}}t_2 - \frac{1}{\tau_2}(t_2 - t_1) = -1, \tag{2}$$

where t_1 stands for the MFPT starting from state 1 at position $\mathbf{r}$, and t_2 for the MFPT starting from state 2 at position $\mathbf{r}$ with velocity $\mathbf{v}$. $\mathbf{I}_a(\mathbf{r})=1$ if $|\mathbf{r}|\leq a$ and $\mathbf{I}_a(\mathbf{r})=0$ if $|\mathbf{r}|>a$. In the present form, these integro-differential equations do not seem to allow for an exact resolution with standard methods. We propose here an approximate resolution based on the introduction of the following auxiliary functions:

$$s(\mathbf{r}) = \frac{1}{2\pi}\int_0^{2\pi} t_2 d\theta_{\mathbf{v}}, \quad \mathbf{d}(\mathbf{r}) = \frac{1}{2\pi}\int_0^{2\pi} t_2\mathbf{v}d\theta_{\mathbf{v}}. \tag{3}$$

Averaging Eq. (2) and Eq. (2) times $\mathbf{v}$ over $\theta_{\mathbf{v}}$, one successively gets

$$\nabla\cdot\mathbf{d} - \frac{1}{\tau_2}[s(\mathbf{r}) - t_1] = -1, \quad \mathbf{d} = \frac{\tau_2}{2\pi}\int_0^{2\pi}(\mathbf{v}\cdot\nabla t_2)\mathbf{v}d\theta_{\mathbf{v}}, \tag{4}$$

which gives in turn

$$\nabla\cdot\mathbf{d} = \frac{\tau_2}{2\pi}\sum_{i,j}\frac{\partial^2}{\partial x_i\,\partial x_j}\langle v_i v_j t_2\rangle_{\theta_{\mathbf{v}}}, \tag{5}$$

where $\langle\cdot\rangle_{\theta_{\mathbf{v}}}$ stands for the average over $\theta_{\mathbf{v}}$. We now make the following decoupling assumption:

$$\langle v_i v_j t_2\rangle_{\theta_{\mathbf{v}}} \simeq \langle v_i v_j\rangle_{\theta_{\mathbf{v}}}\langle t_2\rangle_{\theta_{\mathbf{v}}} = \frac{v^2}{2}\delta_{ij}s(\mathbf{r}) \tag{6}$$

which leads, together with Eq. (4), to the diffusionlike equation

$$\widetilde{D}\nabla^2 s(\mathbf{r}) - \frac{1}{\tau_2}[s(\mathbf{r}) - t_1] = -1 \tag{7}$$

where $\widetilde{D}=v^2\tau_2/2$. Rewriting Eq. (1) as

$$D\nabla^2 t_1 + \frac{1}{\tau_1}[s(\mathbf{r}) - t_1] - k\mathbf{I}_a(\mathbf{r})t_1 = -1, \tag{8}$$

Eqs. (7) and (8) together with vanishing normal derivatives at $|\mathbf{r}|=b$ provide a closed system for the variables s and t_1, whose resolution is lengthy but standard. The validity domain of assumption (6) is much broader than the "Brownian" limit $v\to\infty$ and $\tau_2\to 0$ with $\widetilde{D}$ fixed, in which t_2 is independent of the direction of $\mathbf{v}$. Indeed, it is also valid in the limit $v\tau_2 \gg b$, in which a ballistic phase includes many reorientations due to successive reflections on the boundary $r=b$. In addition, it can be shown that in one dimension this assumption is exact.

We first present the solution of Eqs. (7) and (8) in the "dynamic mode" ($k\to\infty$). The search time $\langle t\rangle$, defined as t_1 uniformly averaged over the initial position of the searcher (note that this last averaging reflects the complete ignorance of the target position), reads in this case

$$\langle t\rangle = (\tau_1+\tau_2)\frac{1-a^2/b^2}{(\alpha^2 D\tau_1)^2}\left\{a\alpha(b^2/a^2-1)\frac{M}{2L_+} - \frac{L_-}{L_+} - \frac{\alpha^2 D\tau_1}{8\widetilde{D}\tau_2}\frac{[3-4\ln(b/a)]b^4 - 4a^2b^2 + a^4}{b^2-a^2}\right\} \tag{9}$$

with

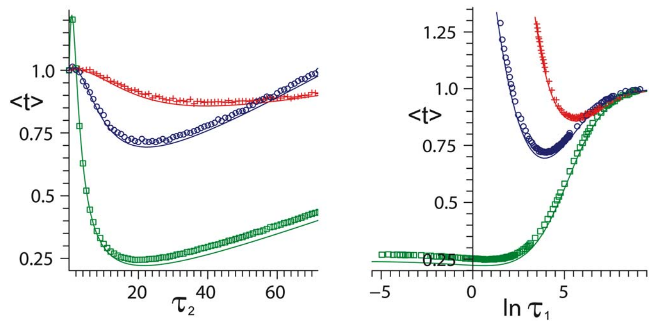

FIG. 2. (Color online) Simulations (points) versus analytical approximate (line) of the search time in the dynamic mode: the search time rescaled by the value in absence of intermittence as a function of τ_2 (left) and ln τ_1 (right) (the logarithmic scale has been used due to the flatness of the minimum), for $D=1$, $v=1$, $b=451$. Left: $a=10$, $\tau_1=1.88$ (green squares); $a=1$, $\tau_1=50$ (blue circles); $a=0.1$, $\tau_1=250$ (red crosses). Right: $a=10$, $\tau_2=21.6$ (green squares); $a=1$, $\tau_2=22.5$ (blue circles); $a=0.1$, $\tau_2=41.6$ (red crosses).

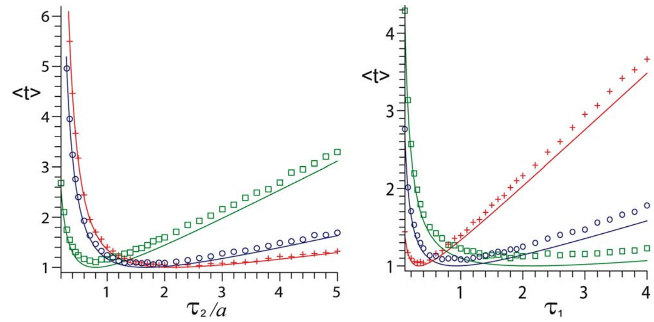

FIG. 3. (Color online) Simulations (points) versus analytical approximate (line) of the search time in the static mode: the search time rescaled by the optimal value as a function of τ_2/a (left) and τ_1 (right) for $k=1$, $v=1$, $b=28$. Left: $a=10$, $\tau_1=2.14$ (green squares); $a=1$, $\tau_1=0.9$ (blue circles); $a=0.1$, $\tau_1=0.3$ (red crosses). Right: $a=10$, $\tau_2=8$ (green squares); $a=1$, $\tau_2=1.7$ (blue circles); $a=0.1$, $\tau_2=0.22$ (red crosses).

$$L_\pm = \mathbf{I}_0(a/\sqrt{\tilde{D}\tau_2})[\mathbf{I}_1(b\alpha)\mathbf{K}_1(a\alpha) - \mathbf{I}_1(a\alpha)\mathbf{K}_1(b\alpha)] \pm \alpha\sqrt{\tilde{D}\tau_2}\mathbf{I}_1(a/\sqrt{\tilde{D}\tau_2})[\mathbf{I}_1(b\alpha)\mathbf{K}_0(a\alpha) + \mathbf{I}_0(a\alpha)\mathbf{K}_1(b\alpha)]$$

and

$$M = \mathbf{I}_0(a/\sqrt{\tilde{D}\tau_2})[\mathbf{I}_1(b\alpha)\mathbf{K}_0(a\alpha) + \mathbf{I}_0(a\alpha)\mathbf{K}_1(b\alpha)] - 4\frac{a^2\sqrt{\tilde{D}\tau_2}}{\alpha(b^2-a^2)^2}\mathbf{I}_1(a/\sqrt{\tilde{D}\tau_2})[\mathbf{I}_1(b\alpha)\mathbf{K}_1(a\alpha) - \mathbf{I}_1(a\alpha)\mathbf{K}_1(b\alpha)],$$

where $\alpha=[1/(D\tau_1)+1/(\tilde{D}\tau_2)]^{1/2}$, and $\mathbf{I}_i$ and $\mathbf{K}_i$ are modified Bessel functions. This expression (9) has proved to be in very good agreement with numerical simulations for a wide range of the parameters (see Fig. 2). The optimization of the explicit expression (9) leads to simple forms in the following situations, depending on the relative magnitude of the three characteristic lengths of the problem $a, b, D/v$. We limit ourselves to the case of low-target density ($a \ll b$), which is the most relevant for hidden target search problems. Three regimes then arise. In the first regime $a \ll b \ll D/v$, the relocating phases are not efficient and intermittence is useless. In the second regime $a \ll D/v \ll b$, it can be shown that the intermittence can significantly speed up the search (typically by a factor 2), but that it does not change the order of magnitude of the search time. On the contrary, in the last regime $D/v \ll a \ll b$, the optimal strategy, obtained for

$$\tau_{1,\min} \sim \frac{D}{2v^2}\frac{\ln^2(b/a)}{2\ln(b/a)-1}, \quad \tau_{2,\min} \sim \frac{a}{v}[\ln(b/a) - 1/2]^{1/2}, \tag{10}$$

leads to a qualitative change of the search time which can be rendered arbitrarily smaller than the nonintermittent search time when $v \to \infty$. This optimal strategy corresponds to a scaling law

$$\frac{\tau_{1,\min}}{\tau_{2,\min}^2} \sim \frac{D}{a^2}\frac{1}{[2-1/\ln(b/a)]^2} \tag{11}$$

which here does not depend on v.

We now turn to the "static mode" ($D \to 0$), which leads to the following expression for the search time:

$$\langle t\rangle = \frac{\tau_1+\tau_2}{2k\tau_1 y^2}\left\{\frac{1}{x}(1+k\tau_1)(y^2-x^2)^2\frac{\mathbf{I}_0(x)}{\mathbf{I}_1(x)} + \frac{1}{4}\{8y^2 + (1+k\tau_1) \times[4y^4\ln(y/x) + (y^2-x^2)(x^2-3y^2+8)]\}\right\}, \tag{12}$$

where

$$x = \sqrt{\frac{2k\tau_1}{1+k\tau_1}}\frac{a}{v\tau_2} \quad \text{and} \quad y = \sqrt{\frac{2k\tau_1}{1+k\tau_1}}\frac{b}{v\tau_2}. \tag{13}$$

Here again, this expression (12) is in very good agreement with numerical simulations for a wide range of the parameters (see Fig. 3). In that case, intermittence is trivially necessary to find the target, and the optimization of the search time (12) leads for $b \gg a$ to

$$\tau_{1,\min} = \left(\frac{a}{vk}\right)^{1/2}\left(\frac{2\ln(b/a)-1}{8}\right)^{1/4}, \tag{14}$$

$$\tau_{2,\min} = \frac{a}{v}[\ln(b/a) - 1/2]^{1/2}, \tag{15}$$

which corresponds to the scaling law $\tau_{2,\min}=2k\tau_{1,\min}^2$, which still does not depend on v.

The main results (10), (14), and (15) obtained in the two modes of search lead us to extract the following remarkable characteristics of intermittent-search processes: (i) In both cases the search time $\langle t\rangle$ presents a global minimum for finite values of the τ_i, which means that intermittence is an optimal strategy. (ii) A very striking and nonintuitive feature is that both modes of search lead to the *same optimal value* of $\tau_{2,\min}$. As this optimal time does not depend on the specific characteristics D and k of the search mode, it seems to constitute a general property of intermittent-search strategies. (iii) The optimal $\tau_{1,\min}$ are different and depend explicitly on D and k, leading to different scaling laws which are susceptible to discrimination between the two search modes.

Finally we remark that this model provides, as a by-product, an approximation for the MFPT for a Pearson-type random walk in the spherical geometry previously defined. The searcher performs ballistic flights reoriented at exponentially distributed times, and, as opposed to standard Pearson walks, the target can be found only when the distance between the target and a reorientation point is less than a. This quantity, obtained here straightforwardly by taking $k \to \infty$ and $\tau_1 \to 0$ in Eq. (12), is written as

$$\langle t \rangle = \frac{(b^2-a^2)^2}{\sqrt{2}vab^2}\frac{\mathbf{I}_0(a\sqrt{2}/v\tau_2)}{\mathbf{I}_1(a\sqrt{2}/v\tau_2)} + \frac{1}{v^2\tau_2 b^2} \times \left[b^4 \ln(b/a) + \frac{1}{4}(b^2-a^2)(a^2-3b^2+4v^2\tau_2^2) \right]. \tag{16}$$

To our knowledge, a similar result for standard Pearson walks is still missing. Note that in the limit $v \to \infty$, $\tau_2 \to 0$ with $\widetilde{D} = v^2\tau_2/2$ fixed, the approximate expression (16) gives back the well-known exact expression for the MFPT of a Brownian particle between concentric spheres [27]. Moreover, for $b \gg a$, the search time (16) is minimized again for the same value (10) and (15) of τ_2, in agreement with the limit $k \to \infty$ of Eqs. (14) and (15).

To conclude, we have proposed a two-state model of search processes for nonrevisitable targets, which closely relies on the experimentally observed intermittent strategies adopted by foraging animals. Using a decoupling approximation numerically validated, we have shown analytically that in the physically most relevant 2D geometry, intermittent strategies minimize the search time, and, therefore, constitute optimal strategies, as opposed to Lévy flights which are optimal only for revisitable targets. We studied two representative modes of search, and determined which features of the corresponding optimal strategies are robust and do not depend on the specific characteristics of the search mode. In particular, both modes lead to a global minimum of the search time as a function of the typical times spent in each state, and the optimal duration of the ballistic relocation phase is the same for both these modes. As this last time does not depend on the nature of the search mode, it could be a universal feature of 2D intermittent search strategies.

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